On Selberg's Central Limit Theorem for Dirichlet L-functions

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Southern Regional Number Theory Conference

The Riemann Zeta Function

For $\Re(s) > 1$, the Riemann zeta function is defined as follows:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} (1 - p^{-s})^{-1}.$$

For $\Re(s) > 0$, one has an integral representation:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx,$$

where $\{x\} = x - [x]$.

The Riemann Zeta Function

Define $\xi(s) := s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. $\xi(s)$ is entire and satisfies $\xi(1-s) = \xi(s)$.

Theorem (Prime Number Theorem)

Let $\pi(x)$ denote the number of primes $p \leq x$.

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

(It follows from the non-vanishing of $\zeta(s)$ on $\Re(s) \geq 1$.)

Conjecture (Riemann Hypothesis)

All the non-trivial zeros of $\zeta(s)$ lie on $\Re(s) = \frac{1}{2}$.

Dirichlet L-functions

Let $q > 1$ be an integer. A Dirichlet character χ modulo q is a homomorphism from $(\mathbb{Z}/q\mathbb{Z})^\times$ to $\mathbb{C}^\times$, extended to $\mathbb{Z}^+$ by setting $\chi(n) = 0$ for $(n, q) > 1$. The Dirichlet L-function attached to χ is defined by

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1}$$

for $\Re(s) > 1$. Let $\chi_0(n) \equiv 1$. We define

$$L(s, \chi_0) = \sum_{n=1}^{\infty} \frac{\chi_0(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s).$$

Dirichlet L-functions

Let $S(t) := \sum_{n \leq t} \chi(n)$. $L(s, \chi)$ also admits an integral representation for $\Re(s) > 0$:

$$L(s, \chi) = s \int_1^\infty \frac{S(t)}{t^{s+1}} dt.$$

Functional equation: $\xi(s, \chi) = \omega_\chi \xi(1 - s, \bar{\chi})$.

Theorem (Dirichlet Theorem on Arithmetic Progressions)

Let $\pi(x; q, a)$ denote the number primes $p \leq x$ such that $p \equiv a$ (mod q). If gcd $(a, q) = 1$, then

$$\pi(x; q, a) \sim \frac{1}{\phi(q)} \frac{x}{\log x}, \quad as \quad x \to \infty,$$

where $\phi(q)$ is the Euler function.

Selberg's Central Limit Theorem

- Selberg (1946) : $\log \zeta(\frac{1}{2} + it)$ is an "approximately" complex normal distribution.
- Selberg (1946) : $\log L(\frac{1}{2} + it, \chi)$ is an "approximately" complex normal distribution.
- Radziwiłł-Soundararajan (2017) : $\log |\zeta(\frac{1}{2} + it)| \sim \mathcal{N}(0, \frac{1}{2} \log \log |t|)$ (New proof).

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X is of $\mathcal{N}(0,\sigma^2)$ if $\mathcal{P}\{\omega \in \Omega : X(\omega) \geq v\sigma\} = \frac{1}{\sqrt{2\pi}} \int_v^\infty e^{-\frac{x^2}{2}} dx$.

Theorem (Selberg)

Let V be a fixed positive real number. Then as $T \to \infty$, one has

$$\frac{1}{T}\mathfrak{L}\Big\{t \in [T, 2T] : \log|\zeta(\frac{1}{2} + it)| \geq v\sqrt{\frac{1}{2}\log\log T}\Big\}$$
$$\sim \frac{1}{\sqrt{2\pi}} \int_v^\infty e^{-\frac{x^2}{2}} dx,$$

uniformly for $v \in [-V, V]$, where $\mathfrak{L}$ denotes the usual Lebesgue measure.

SCLT for Dirichlet L-functions

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Let χ be a primitive Dirichlet character and V a fixed positive real number. Then as $T\to\infty$, one has

$$\frac{1}{T}\mathfrak{L}\Big\{t\in[T,2T] : \log|L(\frac{1}{2}+it,\chi)|\geq v\sqrt{\frac{1}{2}\log\log T}\Big\}$$
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A New Proof of SCLT for Dirichlet L-functions

Key ingredient:

- Multiplicativity of $\chi(n)$.
- The uniform upper bound of χ, i.e., $|\chi(n)| \leq 1$.
- Method of Moments. (characterizng normal distribution by its moments.)
- Approximate functional equation for $L(s, \chi)$.

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Remark

- We only need a standard zero estimate: Let χ be a primitive Dirichlet character (mod q) and $N(T,\chi)$ be the number of zeros of $L(\sigma + it, \chi)$, in the rectangle $0 < \sigma < 1$, $|t| < T$. We have

$$N(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2\pi} - \frac{T}{2\pi} + O(\log qT), \text{ for } T \geq 2.$$

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Independence Property

$\log|L(\frac{1}{2}+it,\chi_1)| \sim \mathcal{N}(0, \frac{1}{2}\log\log|t|)$

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Are $\log|L(\frac{1}{2}+it,\chi_1)|$ and $\log|L(\frac{1}{2}+it,\chi_2)|$ independent variables?

Independence: the outcome of one thing would not effect the outcome of the other one.

- Suppose X and Y are independent random variables. Then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.
- Let X be of $\mathcal{N}(0,\sigma_x^2)$ and Y of $\mathcal{N}(0,\sigma_y^2)$. If they are independent, then $X+Y$ is of $\mathcal{N}(0,\sigma_x^2+\sigma_y^2)$. (Basically, the reverse is correct.)

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Independence Property

Theorem (H.-Wong, 2019)

Let χ_1 and χ_2 be distinct primitive Dirichlet characters. For T sufficiently large and $t \in [T, 2T]$, the random vector

$$\big(\log|L(\frac{1}{2}+it,\chi_1)|, \log|L(\frac{1}{2}+it,\chi_2)|\big)$$

is, approximately, a bivariate normal distribution with mean vector 0_2 and covariance matrix $\frac{1}{2}(\log\log T)I_2$.
Consequently, the random variables $\log|L(\frac{1}{2}+it,\chi_1)|$ and $\log|L(\frac{1}{2}+it,\chi_2)|$ are approximately independent. In particular, for any non-trivial Dirichlet character χ, $\log|\zeta(\frac{1}{2}+it)|$ and $\log|L(\frac{1}{2}+it,\chi)|$ are approximately independent.

Independence Property

Proposition (H.-Wong, 2019)

Let χ_1 *and* χ_2 *be distinct primitive Dirichlet characters. Let* V *be a fixed positive real number. As* $T \to \infty$, *we have, for* $a_1, a_2 \in \mathbb{R}$,

$$\frac{1}{T}\mathfrak{L}\left\{t \in [T, 2T] : \log|\mathcal{L}_{a_1,a_2}(\tfrac{1}{2}+it)| \geq v\sqrt{\frac{a_1^2+a_2^2}{2}\log\log T}\right\} \sim \frac{1}{\sqrt{2\pi}}\int_v^\infty e^{-\frac{x^2}{2}}\,dx,$$

uniformly in $v \in [-V, V]$, *where*

$$\mathcal{L}_{a_1,a_2}(s) = \mathcal{L}(s, \chi_1, \chi_2; a_1, a_2) := |L(s,\chi_1)|^{a_1}|L(s,\chi_2)|^{a_2}.$$

In other words, $\log|\mathcal{L}_{a_1,a_2}(\frac{1}{2}+it)| \sim \mathcal{N}(0, \frac{a_1^2+a_2^2}{2}\log\log T)$.

Independence Property

Remark

- Selberg (1989) did remark "statistical independence." It seems that (at least, according to the argument sketched in the article and the language of modern probability theory) Selberg's assertion is more close to the "uncorrelatedness" among random variables, which is a consequence of the "independence." $(a_1 = a_2 = 1)$
- Introducing an additive structure$\Rightarrow$Breaking multiplicativity!

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Gaussian Process

Theorem (H.-Wong, 2019)

Let $(\chi_j)_{j=1}^{n}$ be a sequence of distinct primitive Dirichlet characters. Then, for T sufficiently large and $t \in [T, 2T]$, the random vector

$$\left(\log |L(\frac{1}{2} + it, \chi_1)|, \cdots, \log |L(\frac{1}{2} + it, \chi_n)| \right)$$

is approximately an n-variate normal distribution with mean vector 0_n and covariance matrix $\frac{1}{2}(\log \log T) I_n$.
Consequently, the random variables $\log |L(\frac{1}{2} + it, \chi_j)|$'s are approximately independent, and $(\log |L(\frac{1}{2} + it, \chi)|)_{\chi \in J}$ forms a Gaussian process for any totally ordered set J of (distinct) primitive Dirichlet characters.

$\mathcal{L}_{a_1, \cdots, a_n}(s) = |L(\frac{1}{2} + it, \chi_1)|^{a_1} \cdots |L(\frac{1}{2} + it, \chi_n)|^{a_n}$ is too hard to handle $\Rightarrow$ Making use of probability theory.

Goal: $\log|L(\frac{1}{2}+it,\chi)| \sim \mathcal{N}(0, \log\log T)$.

Sketch of proof:

- For $\sigma > \frac{1}{2}$, $\log|L(\frac{1}{2}+it,\chi)|$ and $\log|L(\sigma+it,\chi)|$ are close. (Allowing to study the problem away from the critical line.)
- Define $P(s) := \sum_{2\leq n\leq X} \frac{\Lambda(n)\chi(n)}{n^s}$. $\Re(P) \sim \mathcal{N}(0, \frac{1}{2}\log\log T)$. (Note $\log L(s,\chi) = \sum_{n=2}^{\infty} \frac{\Lambda(n)\chi(n)}{n^s}$)
- Define $M(s) := \sum_n \frac{\mu(n)a(n)\chi(n)}{n^s}$. $M(s) \approx e^{P(s)}$.
- $L(s,\chi) \approx M^{-1}(s)$. (Note $L^{-1}(s,\chi) = \sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s}$)

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Proof of Independence Property

Goal:

$\log\left(|L(\frac{1}{2}+it,\chi_1)|^{a_1}|L(\frac{1}{2}+it,\chi_2)|^{a_2}\right) \sim \mathcal{N}(0, \frac{a_1^2+a_2^2}{2}\log\log T)$

Sketch of proof (basically follows the proof of SCLT):

- $\log\left(|L(\frac{1}{2}+it,\chi_1)|^{a_1}|L(\frac{1}{2}+it,\chi_2)|^{a_2}\right) =$
  $a_1\log|L(\frac{1}{2}+it,\chi_1)| + a_2\log|L(\frac{1}{2}+it,\chi_2)|$
- For $\sigma > \frac{1}{2}$, $\log|L(\frac{1}{2}+it,\chi_k)|$ and $\log|L(\sigma+it,\chi_k)|$ are close.
- The corresponding series in moment calculation is
  $P_{a_1,a_2}(s) = \sum_{2\le n\le X}\frac{\Lambda(n)[a_1\chi_1(n)+a_2\chi_2(n)]}{n^s}$.
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Thank you.